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Rationality of Parameterizing Varieties for Modules Over Finite-Dimensional Algebras

Author
Saritzky, Nathan

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Rationality of Parameterizing Varieties for Modules Over Finite-Dimensional Algebras

A thesis submitted in partial satisfaction of the requirements for the degree Master of Arts in Mathematics

by Nathan Saritzky

Committee in charge:

Professor Birge Huisgen-Zimmermann, Chair

Professor Kenneth Goodearl

Professor Jon McCammond

December 2014

The Thesis of
Nathan H. Saritzky is approved:

Professor Kenneth Goodearl

Professor Jon McCammond

Professor Birge Huisgen-Zimmermann, Committee Chairperson

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Abstract

Rationality of Parameterizing Varieties for Modules Over Finite-Dimensional Algebras

by

Nathan Saritzky

One can use classical varieties to attack the problem of classifying finitely-generated modules over finite-dimensional algebras. Given such an algebra, one can write down a number of varieties which parameterize modules with certain isomorphism invariants. Furthermore, these varieties come with morphic actions by algebraic groups whose orbits are in one-to-one correspondence with isomorphism classes of such modules. Using path algebras modulo relations, we can exploit the quiver structure to learn about the structure of these varieties. We use this to give a proof of rationality of one such variety parameterizing graded modules.

Contents

Introduction

It is the algebraist's favorite activity to classify things. In any field in algebra, the driving problem is the classification of all objects in question, be they groups, rings, fields, modules, varieties, maps among all these objects, etc. The game is to be able to write down a comprehensible list of invariants which 1.) allow one to answer any question one may have about said structure, and 2.) are sufficient to describe every algebraic structure in the Universe. When this turns out to be an unreasonable goal, one cuts down, and down, and down, until one reaches a class of objects understandable within a human lifetime.

This is the general program of this thesis.

Our objects in question are finitely generated modules over a finite-dimensional algebra. Such modules come with strong finiteness properties yielding fundamentally useful tools, such as a composition series and a unique direct sum decomposition. These help prevent us from being totally lost, as one might be in a more general class of modules. However, the task of classifying all such modules is still not feasible in general, at least if one tries to do so in a unified framework. One needs to subdivide the task to open up accessible portions.

Our central focus will be path algebras modulo relations. First defined by Gabriel [7], they are a class of algebras which can tell us any story told by module categories over finite-dimensional algebras. More precisely, if A is any finite dimensional algebra over an algebraically closed field, then there exists a unique (up to isomorphism) path algebra modulo relations Λ, together with an additive equivalence of categories from A-Mod to Λ-Mod. Moreover, path algebras modulo relations admit combinatorial and visual approaches making them more accessible.

To classify the modules over our algebra Λ, we employ various parameterizing varieties. The first is $\mathbf{Mod}_d(\Lambda)$, an affine variety parameterizing all isomorphism classes of left Λ modules with dimension d. The variety $\mathbf{Mod}_d(\Lambda)$ carries a $\mathrm{Gl}_d(k)$-action, the orbits of which are in bijective correspondence with these isomorphism classes. If $\mathbf{Mod}_d(\Lambda)$ had a geometric quotient modulo this group action, it would yield classifying isomorphism invariants of the corresponding modules. (Roughly, a geometric quotient, if it exists, is the topological quotient under the Zariski topology, endowed with a structure of an algebraic variety linking it to the Gl_d-space $\mathbf{Mod}_d(\Lambda)$ by way of a strong universal property. But existence fails badly unless the algebra Λ is semisimple.)

To gain insight, we define projective and quasiprojective parameterizing varieties with strong geometric ties to $\mathbf{Mod}_d(\Lambda)$. We start with a semisimple left Λ-module T and P its distinguished projective cover (see Corollary 3.10). If J is the Jacobson radical of Λ and $\mathrm{Gr}(m, V)$ the classical grassmannian of m-dimensional vector subspaces of a vector space V, we define a projective variety

$$\mathfrak{Grass}_d^T = \{C \in \mathrm{Gr}(\dim_k JP - d, JP) \mid C \text{ is a } \Lambda\text{-submodule of } JP\}.$$

This parameterizes left Λ-modules M of dimension d with top T, i.e. those for which $M/JM \cong T$, by way of the assignment $C \mapsto P/C$. It is then cut further in the following way: The *radical layering* of M is the sequence $(M/JM, JM/J^2M, \ldots, J^LM/J^{L+1}M)$ of semisimple modules, where L is the largest natural number for which $J^L \neq 0$. Let $\mathbb{S} = (\mathbb{S}_0, \ldots, \mathbb{S}_L)$ be a sequence of semisimple left Λ-modules. We then define

$$\mathfrak{Grass}(\mathbb{S}) = \{C \in \mathfrak{Grass}_d^T \mid C \text{ has radical layering } \mathbb{S}\}.$$

This quasiprojective variety has a couple of substantial benefits. The first is that it tends to have considerably smaller dimension that its affine cousin within $\mathbf{Mod}_d(\Lambda)$. Second is a useful open affine cover.

This thesis is particularly concerned with the setting of *graded* modules. If Λ is a graded k-algebra, and J and P are as above, we define

$$\text{Gr-}\mathfrak{Grass}(\mathbb{S}) = \{C \in \mathfrak{Grass}(\mathbb{S}) \mid C \text{ is a homogeneous submodule of } JP\}.$$

The quotient modules P/C are then graded, whence Gr-$\mathfrak{Grass}(\mathbb{S})$ parameterizes the *graded* left Λ modules with radical layering $\mathbb{S}$.

We will define the *skeleton* of a module M, which is a particular class of vector-space basis. The advantages of skeleta are 1.) they give us the just-mentioned open affine cover of $\mathfrak{Grass}(\mathbb{S})$ and Gr-$\mathfrak{Grass}(\mathbb{S})$, and 2.) one can actually compute them. A skeleton may be visualized as a forest of rooted trees. Skeleta will be used to prove the central theorem of this thesis, which was stated in general and proved for the ungraded setting by Babson, Huisgen-Zimmermann, and Thomas in [3], Theorem 5.3.

Theorem. Let $\Lambda = kQ/I$ be a path algebra modulo relations, where I is the ideal generated by all paths in Q of length $L + 1$ for some natural number L. If Gr-$\mathfrak{Grass}(\mathbb{S})$ is nonempty, then it is rational, irreducible, and smooth.

1 Algebraic Preliminaries

1.1 Setting the Stage

In the section, we briefly go over concepts from the theory of finite-dimensional algebras we will need. Much of what is discussed here will have a more intuitive or combinatorial interpretation once we have specialized to path algebras modulo relations. For the most part, we will not prove results unless said proofs are particularly relevant to the rest of the thesis.

First, we establish some basic terminology and notation. All our rings are unital. An algebra over a field k is a ring A together with an injective ring homomorphism of k into the center of A. It follows from this definition that A is a k-vector space, and its vector space structure “plays nicely” with the ring structure. Throughout, A will be a finite-dimensional algebra over an algebraically closed field k.

First, we observe that A is then both noetherian and artinian. This follows from the fact that any properly ascending or descending chain of ideals will be, in particular, a chain of k-subspaces of A.

If M is a left A-module, M is left noetherian (artinian) if it satisfies the ascending (descending) chain condition for submodules. If M is both left noetherian and left artinian, we say that M has finite length.

We write A-Mod for the category of left A-modules, and A-mod (note the lower-case “m”) for the full subcategory of *finitely generated* left A-modules. Equivalently, A-mod is the category of left A-modules which are finite-dimensional k-vector spaces.

Definition 1.1. A *composition series* for $M \in A$-Mod is an ascending chain of submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_c = M$$

such that M_j/M_{j-1} is simple for $1 \leq j \leq c$. The M_j/M_{j-1} are called the *composition factors* of the series.

For modules of finite length, one has existence and uniqueness (in a sense to be described in the following theorem) of composition series:

Theorem 1.2 (Jordan-Hölder)**.** Let $M \in A$-Mod be a module of finite length (such as when M is finitely generated). The M has a composition series. Moreover, it is unique in the following sense: If we have two composition series

$$0 = M_0 \subsetneq \cdots \subsetneq M_c = M, \quad 0 = N_0 \subsetneq \cdots \subsetneq N_d = M,$$

then $c = d$, and there is a permutation σ of the index set $\{1, \ldots, c\}$ such that

$$M_j/M_{j-1} \cong N_{\sigma(j)}/N_{\sigma(j)-1}.$$

That is, the composition factors are unique "up to shuffling".

Proof. See [1], Theorem 11.3. □

Definition 1.3. In light of the Jordan Hölder theorem, we call c the *composition length* of M.

The existence and uniqueness of this composition series is a very strong and useful property. One can use it to prove the essential fact that there exist a unique decomposition of M into indecomposable summands.

Theorem 1.4 (Krull-Remak-Schmidt)**.** Let $M \in A$-mod be a module of finite length. Then there exist indecomposible submodules $M_1, \ldots, M_r$ such that $M = \bigoplus_{1 \leq i \leq r} M_i$. Moreover, if we have another such decomposition $M = \bigoplus_{1 \leq i \leq s} N_i$, then $r = s$, and there is a permutation σ of the index set $\{1, \ldots, r\}$ such that $M_i \cong N_{\sigma(i)}$ for each i.

Proof. See [1], Theorem 12.9. □

Note that this is the strongest such result one could possibly hope for. In particular, isomorphism of the indecomposable summands, rather than equality, is the best one could reasonably expect. For example, one has the following two decompositions of the vector space k^2 into indecomposable summands whenever $\operatorname{char} k \neq 2$:

$$k^2 = k(1, 0) \oplus k(0, 1) = k(1, 1) \oplus k(-1, 1).$$

Krull-Remak-Schmidt tells us that if we understand the indecomposable modules in A-mod, then we understand all modules, for they will, essentially uniquely, be finite direct sums of those indecomposables. The task of understanding the finitely-generated left A

modules may therefore be reduced to understanding the finitely generated *indecomposable* modules. In pursuit of this goal, it is then natural to consider the question of “how many” indecomposable left A-modules we have.

The following is a somewhat sketchy definition, intended to motivate rather than be used seriously:

Definition 1.5.
- An algebra A is called *tame* if all the isomorphism classes of indecomposible left A-modules of fixed dimension beling to one of a finite number of “1-parameter families” together with finitely many “sporadic” isomorphism classes.
- A is called *wild* if there is an equivalence of categories from $k\langle x, y\rangle$-mod to a full subcategory of A-mod.

Tame algebras, as their name suggests, have comparatively easy representation theory. One can, in principle, write down a complete list of the indecomposables in an understandable format. Wild algebras, on the other hand, have representation theory “as bad” as that of $k\langle x, y\rangle$, which could fairly be described as “awful”, “brain-meltingly difficult”, or “quite possibly forever beyond human understanding”. In this instance, describing the indecomposable modules all in one go is essentially a hopeless endeavor. The remarkable fact here is that there is no middle ground in our setting. The representation theory of A has only two settings: “quite nice” and “terrible”.

Theorem 1.6 (Drozd Dichotomy, see [5])**.** Let A be a finite-dimensional algebra. Then A is either tame or wild.

More often than not, one lands on the wild side of this divide. This leads us to, rather than abandoning every hope, the goal of “cutting down” our category into more manageable pieces. It is here where our work will be done.

1.2 Graded Algebras

Definitions 1.7. 1. We say that an algebra A is *graded* if there exists a vector space decomposition $A = \bigoplus_{m \in \mathbb{N} \cup \{0\}} A_m$ such that if $a \in A_m$ and $b \in A_l$, then $ab \in A_{m+l}$. In this case, a left A-module M is graded if M has a vector space decomposition $M = \bigoplus_{m \in \mathbb{N} \cup \{0\}}$ such that if $a \in A_m$ and $b \in M_l$, then $ab \in M_{m+l}$.

2. If $a \in A_l$, we say that a is *homogeneous of degree* l. We say the same of an element $b \in M_l$.

3. We call a submodule $C \subseteq M$ *homogeneous* if it is generated by homogeneous elements (of any degrees).

For a submodule C of a graded module M, the quotient module M/C inherits the grading from M if and only if C is homogeneous.

1.3 Semisimple Modules and the Radical

Were dreams always to come true in representation theory, we would be able to break down every left A-module into not just indecomposable, but *simple* left A-modules[1] This is just the case in semisimple algebras, which have the "nicest" (read: "most boring") representation theory.

Definition 1.8. A left A-module M is *semisimple* if the following equivalent conditions hold:

1. M is the sum of its simple submodules.
2. M is a *direct* sum of simple submodules.
3. Every submodule of M is a direct summand.

A itself is called semisimple if the left regular module ${}_AA$ is semisimple.

[1] Of course, this would leave those who study representations of algebras with rather a lot less to do.

If A is a semisimple ring, then one has that every $M \in A$-mod is a semisimple module. In this case, the representation theory of A is relatively straightforward: so long as one understands the simple left A-modules, one understands all finitely-generated left A-modules. Fortunately, the simple modules are typically easier to understand than the indecomposables. In fact, in our setting, we will understand the simple modules very well.

However, it is far too much to hope that our algebras are always semisimple. But the culprit is easily identified, and may be modded out to restore some order:

Definition 1.9. Let A be a finite-dimensional algebra. The Jacobson radical J of A is the intersection of all maximal left-ideals.

One can equivalently define the Jacobson radical of A to be the intersection of all maximal *right*-ideals of A. In particular, it is a two-sided ideal.

Notation. Throughout this thesis, J will be reserved for referring to the Jacobson radical of the ring in question.

Your favorite text on ring theory (e.g. [1]) will list myriad characterizations and properties of the Jacobson radical. For our purposes, we will need the following (which hold for artinian rings, but not in general):

Proposition 1.10. Let A be a finite-dimensional algebra (or, more generally, an Artinian ring).

1. A/J is a semisimple ring.

2. J is unique largest nilpotent ideal of A.

3. J annihilates every simple left A-module. In particular, every simple left A-module is also a left A/J-module.

4. Let $M \in A$-mod. Then JM is the intersection of all the maximal submodules of M, and M/JM is semisimple.

Observation and Definition 1.11. Let $M \in A$-mod. For any natural number l, one sees from Proposition 1.10(4) that $J^l M / J^{l+1} M$ is semisimple. Moreover, the nilpotence of J gives us a unique smallest natural number L such that $J^L M = 0$. The collection of semisimple modules $M/JM, JM/J^2M, \ldots, J^{L-1}M/J^L M$ is called the *radical layering of* M. In particular, M/JM is referred to as the *top* of M.

The notion of the radical layering allows us to leverage the comparatively easy-to-understand simple left A-modules to peer into the general representation theory of A. Path algebras, we will see, provide us with a way to visualize how the radical layers of A "link up".

1.4 The Jacobson Radical and Projective Covers

Projective covers of modules are going to be an essential tool for us. They have a few different characterizations, but, in our setting (that is, finitely-generated modules over artinian rings), they have a relatively concrete description utilizing the Jacobson radical. Here we will present this description rather than something more general.

Definition 1.12. A left A-module P is *projective* if, for any left A-modules M and N, with a homomorphism $f : P \to M$ and surjective homomorphism $g : N \twoheadrightarrow M$, there exists a homomorphism $h : P \to N$ satisfying the following commmutative diagram:

$$\begin{tikzcd} & N \arrow[d, "g"] \\ P \arrow[ur, dashed, "h"] \arrow[r, "f"'] & M \end{tikzcd} .$$

In full generality, projective modules have a very concrete characterization: they are direct summands of free modules.

Proposition 1.13. $P \in A$-Mod is projective if and only if there exists $P' \in A$-Mod such that $P \oplus P'$ is a free left A-module.

With this in mind, it is easy to see that every module $M \in A$-mod is the surjective image of a projective module. Indeed, every left A-module is the quotient of some free left A-module. However, it is clear that one could choose some enormous free module and then factor out most of it to get M. Our hope is that we can find a "minimal" projective module P with a homomorphism onto M having "small" kernel. Indeed, we have just that, if we define "small" to mean "contained in JP".

Proposition and Definition 1.14. Let $M \in A$-mod.

1. There exists a projective module P together with a surjective homomorphism $f : P \twoheadrightarrow M$ such that $\ker f \subseteq JP$. We call P a *projective cover* of M.

2. If P and P' with surjections f, f', respectively, are projective covers of M, then there exists an isomorphism $g : P \to P'$ making the following diagram commute:

$$\begin{tikzcd} P \arrow[dd, "g"', "\sim"] \arrow[dr, two heads, "f"] & \\ & M \\ P'. \arrow[ur, two heads, "f'"] & \end{tikzcd}$$

We omit the proof, but we will explicitly construct a particularly useful projective cover in Corollary 2.10.

1.5 Idempotents and Basic Algebras

Recall that an *idempotent* in A is an element e with $e^2 = e$. A pair of idempotents e, f is called *orthogonal* if $ef = fe = 0$. We will call a set of idempotents orthogonal if its elements are pairwise orthogonal. The significance of orthogonal idempotents is demonstrated by the following basic fact in ring theory:

Proposition 1.15. Let R be a ring. Then the following are equivalent for any collection of left ideals $I_1, \ldots, I_n$ of R:

1. ${}_R R = \bigoplus_{1 \le j \le n} I_j$ as left R-modules.
2. There exist orthogonal idempotents $e_1, \ldots, e_n$ such that $I_j = Re_j$ and $1 = e_1 + \cdots + e_n$.

This tells us that breaking down ${}_R R$ into summands is equivalent to finding orthogonal idempotents. In fact, the problem of breaking a ring down into *indecomposable* summands can be stated in terms of idempotents.

Definition 1.16.
1. An nonzero idempotent e is called *primitive* if whenever $e = e' + e''$ for orthogonal idempotents e', e'', then either $e = e'$ or $e = e''$.
2. A set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$ is called a *full sequence* of orthogonal primitive idempotents if $e_1 + \cdots + e_n = 1$ (typically, we will suppress the word "orthogonal").

The following test for primitivity of an idempotent will come in handy:

Lemma 1.17. $e \in A \setminus \{0\}$ is a primitive idempotent if and only if it is the only nonzero idempotent in the ring eAe.

Proof. Suppose that e is primitive and let $f \in eAe$ be an idempotent. Then $e - f$ is an idempotent orthogonal to f, and $e = f + (e - f)$. Primitivity of e then gives that either $f = 0$ or $f = e$. Conversely, suppose that e is the only nonzero idempotent of eAe and that $e = f + f'$ for orthogonal idempotents f, f'. Then Then $ef = f^2 + f'f = f^2 = f$. Similarly, $fe = f$. Therefore $f \in eAe$, and our hypothesis then yields that $f = 0$ or $f = e$, as desired. $\square$

Evidently, if e is an idempotent, then Re is indecomposable if and only if e is primitive. Therefore, expressing a ring as a direct sum of left ideals is equivalent to finding a full sequence of primitive idempotents. Ideally, we would like for our sequence to irredundantly represent the indecomposable summands of A. This brings us to the notion of a basic algebra.

Definition 1.18. A finite-dimensional algebra A is called *basic* if there exists a full sequence of primitive idempotents $\{e_1, \ldots, e_n\}$ such that if $i \neq j$, then $Ae_i \ncong Ae_j$. In this case, we call $\{e_1, \ldots, e_n\}$ a *basic set* of idempotents.

In the following, $E_{i,j}$ is the matrix (of appropriate size) with a 1 in the i,j-entry and zeroes everywhere else.

Example 1.19. The ring T_n of upper triangular n-by-n matrices over k is a basic algebra. The matrix idempotents $E_{1,1}, \ldots, E_{n,n}$ form a full sequence of orthogonal primitive idempotents. $T_nE_{i,i}$ is the ideal of matrices with nonzero entries only in the i^{th} column. One then sees that $\dim_k(T_nE_{i,i}) = i$, whence $T_nE_{i,i} \ncong T_nE_{j,j}$ whenever $i \neq j$.

Non-example 1.20. The ring M_n of all n-by-n matrices over k is *not* a basic algebra for $n \geq 2$. The matrix idempotents $E_{1,1}, \ldots, E_{n,n}$ still form a full sequence of primitive idempotents, but the summands $M_nE_{i,i}$ fail to be pairwise non-isomorphic. In fact, for any $1 \leq i, j \leq n$, the map $f_{i,j} : T_nE_{i,i} \to T_nE_{j,j}$ given by $M \mapsto ME_{i,j}$ is an isomorphism.

While not all algebras are basic, the basic algebras do, on their own, capture "all" of the representation theory of finite-dimensional algebras, at least in the sense of categories.

Definition 1.21. Let R and S be rings. We say that R and S are *Morita equivalent* if there exists an additive equivalence of categories between R-Mod and S-Mod.

The remarkable fact is that Morita equivalence is more or less fully understood. In our setting, we have

Theorem 1.22 (See [2], Theorem 6.8)**.** Every finite-dimensional algebra A over an algebraically closed field k is Morita equivalent to a basic algebra B, and B is unique up to isomorphism.

This thesis is concerned with path algebras modulo relations, which, we will see, are basic algebras. By restricting our attention to these, we examine a canonical set of representatives of the Morita equivalence classes of finite-dimensional algebras over algebraically closed fields. But, lest we get too excited, be warned that Morita equivalence is a highly imperfect partition. For example, one has that the matrix ring $M_n(k)$ is Morita equivalent to k for all n! These, however, are clearly very different rings.

2 Path Algebras and Path Algebras modulo Relations

2.1 A Primer on Quivers

Definition 2.1. A quiver is a finite directed graph in the most general sense. That is, a collection of vertices $V_0 = \{e_1, \ldots, e_n\}$ and arrows V_1, both finite, where each $\alpha \in V_1$ is an arrow from e_i to e_j. We allow for loops (i.e. arrows from e_i to itself) and multiple edges. Given an arrow α, we write $\text{start}(\alpha)$ and $\text{end}(\alpha)$ for the starting and ending vertices of α, respectively. A path is then a finite sequence of arrows $p = \alpha_\ell \alpha_{\ell-1} \cdots \alpha_1$ such that, for each $1 \leq i \leq \ell - 1$, we have $\text{end}(\alpha_i) = \text{start}(\alpha_{i+1})$. We then write $\text{start}(p) = \text{start}(\alpha_1)$ and $\text{end}(p) = \text{end}(\alpha_\ell)$. The integer ℓ is the length of p, written $\ell(p)$. Additionally, we allow for a path of length 0 at each vertex, and identify this path with the vertex itself. A path z is an *oriented cycle* if z has positive length and $\text{start}(z) = \text{end}(z)$.

Given a quiver Q, let kQ be the vector space whose basis consists of all paths in Q. We endow kQ with a multiplication induced by concatenation of paths. That is, if $p = \alpha_\ell \cdots \alpha_1$ and $q = \beta_m \cdots \beta_1$ are paths in Q, then pq (which we take as meaning "p after q") is

$$pq = \begin{cases} \beta_m \cdots \beta_1 \alpha_\ell \cdots \alpha_1 & \text{if } \text{start}(p) = \text{end}(q) \\ 0 & \text{otherwise.} \end{cases}$$

We then extend this multiplication of paths linearly to all of kQ to make it a k-algebra.

Examples 2.2. 1. Let Q be the following quiver:

$$\begin{tikzcd} 1 \arrow[loop above, "\alpha"] \end{tikzcd}.$$

The algebra kQ has identity e_1. Since α begins and ends at the same point, $\alpha^2 \neq 0$. Indeed, α^k is nonzero for any natural number k. The algebra we have then consists of linear combinations of e_1 and positive powers of α. The result is isomorphic to the polynomial ring $k[x]$. An explicit isomorphism is the map induced by $e_1 \mapsto 1$ and $\alpha \mapsto x$.

2. Let Q be the following quiver:

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n.$$

kQ has orthogonal idempotents $\{e_1, \ldots, e_n\}$. Each of the rings $e_i(kQ)e_i$ is the span of the paths starting and ending at the vertex e_i. Of course, there are no such paths aside from e_i itself. It follows that e_i is the only idempotent in $e_i(kQ)e_i$, and so is primitive by Lemma 1.17. Moreover, if we have a path $\alpha_s\alpha_{s-1}\cdots\alpha_t$, where $1 \leq t \leq s \leq n-1$ in Q, we see that $(e_1 + \cdots + e_n)\alpha_s\alpha_{s-1}\cdots\alpha_t = e_s\alpha_s\alpha_{s-1}\cdots\alpha_t = \alpha_s\alpha_{s-1}\cdots\alpha_t$. Therefore $e_1 + \cdots + e_n = 1$, and so $\{e_1, \ldots, e_n\}$ comprises a *full* sequence of primitive orthogonal idempotents.

For each vertex e_i, the indecomposible left kQ-module kQe_i consists of linear combinations of paths starting in e_i, of which there are $n-i$. Therefore $\dim_k(kQe_i) = n-i$. Since the kQe_i's have nonequal dimensions, we have that kQ is a basic algebra. In fact, the ring homomorphism from kQ to the ring T_n of lower-triangular n-by-n matrices induced by $\alpha_s \cdots \alpha_t \mapsto E_{s+1,t}$ is an isomorphism (cf. Example 1.19).

Much of the phenomena exhibited in these examples happens in general.

Observations 2.3. Let Q be a quiver with veretx set $\{e_1, \ldots, e_n\}$.

1. The paths of length zero $e_1, \ldots, e_n$ are orthogonal idempotents.

2. If p is a path, then

$$e_i p = \begin{cases} p & \text{if } \operatorname{end}(p) = e_i \\ 0 & \text{otherwise} \end{cases}, \quad p e_j = \begin{cases} p & \text{if } \operatorname{start}(p) = e_j \\ 0 & \text{otherwise.} \end{cases}$$

3. In light of the above observation, we see that $\sum_{i=1}^{n} e_i$ is a multiplicative identity, and therefore kQ is a unital ring.

4. Each $e_i(kQ)e_i$ is a ring with identity e_i, consisting of linear combinations of e_i and cycles at e_i.

5. The vector space $e_j(kQ)e_i$ has dimension equal to the number of paths from e_i to e_j.

6. kQ can be given a grading. Let kQ_ℓ be the subspace generated by paths of length ℓ. Then kQ has a vector space decomposition

$$kQ = \bigoplus_{\ell \in \mathbb{N} \cup \{0\}} kQ_\ell,$$

which yields a grading on kQ, called the *path length grading.*

7. $\dim_k kQ < \infty$ if and only if Q has no oriented cycles. Indeed, if $z \in kQ$ is an oriented cycle, then $\{z^n\}_{k \in \mathbb{N}}$ is an infinite linearly-independent set in kQ. Conversely, if Q has no oriented cycles, then it has only finitely many paths (due to finiteness of the vertex and arrow sets).

Proposition 2.4. Let Q be a quiver. Then the vertex set $\{e_1, \ldots, e_n\}$ comprises a basic set of primitive idempotents for kQ.

Proof. It's clear from the nature of the multiplication in kQ that the e_is are orthogonal idempotents. In view of Observation 2.3(3), $\sum_{i=1}^{n} e_i = 1$, and $\{e_1, \ldots, e_n\}$ is a *full* sequence of orthogonal idempotents. Finally, suppose that $(kQ)e_i \cong (kQ)e_j$. Then their left annihilator ideals l-Ann$((kQ)e_i)$ and l-Ann$((kQ)e_j)$ are equal. If we suppose further that $i \neq j$, then $e_j e_i = 0$ and so $e_i \in$ l-Ann$((kQ)e_j) =$ l-Ann$((kQ)e_i)$. Hence $0 = e_i^2 = e_i$, a contradiction. Therefore $i = j$.

It remains to be shown that e_i is a primitive idempotent. By Lemma 1.17, it suffices to show that it is the only nonzero idempotent in $e_i(kQ)e_i$. Suppose that e is such an idempotent. Then e has the form

$$e = ae_i + z,$$

where $a \in k$ and z is a linear combination of cycles at e_i. Then

$$0 = e^2 - e = (a^2 - a)e_i + (2a - 1)z + z^2.$$

Hence $z = 0$, and $a^2 - a = 0$. Therefore a is either 0 or 1, yielding $e = 0$ and $e = e_i$, respectively. □

Proposition 2.4 gives that $\bigoplus_{i=1}^{n}(kQ)e_i$ is a decomposition of kQ into indecomposable summands. The indecomposible projective modues $(kQ)e_i$ and their simple quotients, $(kQ)e_i/J_0e_i$, where J_0 is the ideal generated by the arrows, play a crucial role in the representation theory of path algebras.

Notation. Typically, we will write J_0 for the ideal in kQ generated by the arrows.

2.2 Path Algebras Modulo Relations

Definition 2.5. Let $I \leq kQ$ be an ideal. We call I an *admissible ideal* if there exists a natural number $L \geq 2$ such that $J_0^L \subseteq I \subseteq J_0^2$. Equivalently, I is admissible if it satisfies:

- I is generated by linear combinations of paths of length at least two.
- All paths of length L are in I.

Note that if Q is a quiver without oriented cycles, then $I = 0$ is admissible. In this case, the paths in Q have uniformly bounded length, and so the powers of J_0 will "die out" on their own. Conversely, if Q does have oriented cycles, then it is necessary for I to be nonzero for it to be admissible.

Proposition 2.6. Let $I \leq kQ$ bean admissible ideal and $\Lambda = kQ/I$ be a path algebra modulo relations. Then Λ is a basic, finite-dimensional algebra.

Much of this proof mirrors that of Proposition 2.4.

Proof. Finite dimensionality of Λ comes from the fact that $J_0^m \subseteq I$ for some natural number m, and kQ/J_0^m is finite-dimensional. It remains to show that Λ has a basic set of orthogonal idempotents. The obvious candidate turns out to be the correct one: the elements $e_1 + I, \ldots, e_n + I$. The fact that $I \subseteq J_0^2$ implies that each $e_i \notin I$, and so $e_i + I$ is nonzero in Λ. They then inherit orthogonality and idempotent-ness from kQ.

Suppose that $\Lambda(e_i + I) \cong \Lambda(e_j + I)$. Then the left annihilator ideals l-Ann$(\Lambda(e_i + I))$ and l-Ann$(\Lambda(e_j + I))$ are equal. If we suppose further that $i \neq j$, then the orthogonality of e_i and e_j gives that $e_i \in$ l-Ann$(\Lambda(e_i + I))$, whence $e_i^2 = e_i \in I$. But, once again, $I \subseteq J_0^2$, so this cannot happen. Therefore $i = j$.

It remains to show that $\Lambda(e_i + I)$ is an indecomposable left Λ-module for each i. Suppose that that we have a direct sum decomposition $\Lambda e_i = \Lambda e \oplus \Lambda f$ for some nonzero orthogonal idempotents $e, f \in \Lambda(e_i + I)$ such that $e + f + I = e_i + I$. $e = \sum_{1 \leq j \leq n}(a_j e_j + q + I)$, where each $a_j \in k$ and q is a linear combination of paths of nonzero length in kQ. A straightforward computation shows that $ee_i = e_ie = e$, whence $a_j = 0$ for $j \neq i$, and q is in fact a linear combination of cycles at e_i. Then

$$e^2 - e + I = 0$$
$$\Rightarrow (a_i^2 - a_i)e_i + (2a_i - 1)q + q^2 \in I.$$

Since $I \subseteq J_0^2$, we must have $a_i^2 - a_i = 0$, so $a_i = 0$ or 1. If $a_i = 1$, then $e_i - e = -q + I$ is an idempotent. But the admissibility of I gives that $q^m \in J_0^m \subseteq I$ for some natural number m. Therefore $-q + I$ is both an idempotent and a nilpotent in Λ, and so $-q + I = 0$, whence $e = e_i + I$. But then $f = 0$, which is a contradiction. Similarly, if $a_i = 0$, then $e = q + I$ is both an idempotent and a nilpotent, and so $e = 0$, also a contradiction. □

We will typically abbreviate our idempotents $e_i + I \in \Lambda$ as simply e_i.

Corollary 2.7. $\Lambda = \bigoplus_{1 \leq i \leq n} \Lambda e_i$ is "the" (in light of the uniqueness afforded by the Krull-Remak-Schmidt theorem) decomposition of Λ into indecomposable summands..

We now show that we know the indecomposable projective and simple objects in Λ-mod "personally". The former are the left ideals Λe_i, and the latter their quotients $S_i = \Lambda e_i / J e_i$.

Lemma 2.8. Let $\Lambda = kQ/I$ be a path algebra modulo relations and J_0 the ideal generated by the arrows in kQ. Then the image of J_0 in Λ by the quotient map is the Jacobson radical of Λ.

Proof. Λ is a finite-dimensional algebra, and so, by [AF 15.19], its Jacobson radical is the unique largest nilpotent ideal. Now, on the one hand, J_0 is nilpotent, for the admissibility of I gives a natural number L such that $J_0^L \subseteq I$. On the other hand, suppose that $J_0 \subsetneq J$. Then there exists a nonzero linear combination of idempotents $E = \sum_{i=1}^n a_i e_i$ and an element $y \in J_0$ such that $E + y \in J_0$. For any natural number m, we have

$$E^m = \sum_{i=1}^{n} a_i^m e_i.$$

E is a nonzero linear combination, so there is some $a_i \neq 0$. It follows that $e^m \neq 0$, since the e_is are linearly independent. Therefore $(E + y)^m \neq 0$. But J is a nilpotent ideal, and so we have a contradiction. $\square$

From Lemma 2.8, we can see that $\Lambda/J \cong \Lambda e_1/Je_1 \oplus \cdots \oplus \Lambda e_n/Je_n$. From this we get the following proposition:

Proposition 2.9. All simple left Λ-modules have the form $\Lambda e_i/Je_i$ for some vertex e_i.

Proof. Let $S \in \Lambda$-mod be simple. Since J annihilates all simple left Λ-modules, S is also a left Λ/J-module. Pick a nonzero $x \in S$, and let $\mu_x : \Lambda/J \to S$ be the map $\lambda + J \mapsto \lambda x$. Precomposing with the canonical injection $\eta_i : \Lambda e_i/Je_i \hookrightarrow \Lambda/J$ gives us maps

$$\mu_x \circ \eta_i : \Lambda e_i/Je_i \to S.$$

The map μ_x is nonzero, and so there must be some nonzero $\mu_x \circ \eta_i$. Schur's lemma then implies that $S \cong \Lambda e_i/Je_i$ for some vertex e_i as left Λ/J-modules. $\square$

Notation. Forevermore, we shall use S_i to refer to the simple module $\Lambda e_i/Je_i$.

Proposition 2.9 yields the following essential corollary, which tells us that we understand "the" projective covers of modules in Λ-mod.

Corollary 2.10. Let M be a left Λ-module. Then M has a projective cover of the form $P = \sum_{i=1}^{n} (\Lambda e_i)^{t_i}$ for some nonnegative integers $t_1 \ldots t_n$.

Proof. Proposition 2.9 implies that there exist nonnegative integers $t_1, \ldots, t_n$ such that $M/JM \cong S_1^{t_1} \oplus \cdots S_n^{t_n}$. Let $P = (\Lambda e_1)^{t_1} \oplus \cdots \oplus (\Lambda e_n)^{t_n}$. Since each Λe_i is a direct summand of the left regular Λ-module Λ, P is projective. Therefore, there is a map $h : P \to M$ satisfying the following commutative diagram:

$$\begin{tikzcd} & M \arrow[d, "\text{quo}"] \\ P \arrow[ur, dashed, "h"] \arrow[r, "\text{quo}"'] & M/JM. \end{tikzcd}$$

Since $\ker h \subseteq JP$, we have that P is a projective cover of M. $\square$

Definition and Notation 2.11. We call the above P the *distinguished projective cover of* M. It will often be convenient to write it as

$$P = \bigoplus_{1 \le r \le t} \Lambda z_r,$$

where $w = \sum_{i=1}^{n} t_i$, and the set $\{z_1, \ldots, z_w\}$ induces a basis of P/JP. Then each z_r has a unique vertex $e(r)$ such that $e(r)z_r = z_r$. We call $z_1, \ldots, z_t$ a *distinguished sequence of top elements* of P.

2.3 Representations of Quivers

Representations of quivers provide an alternative viewpoint for modules over path algebras modulo relations. They more readily expose explicit ties between path algebras and linear algebra, and are a popular object of study in their own right. In the interest of completeness, we present the relevant definitions and constructions without proofs.

Definitions 2.12. Let Q be a quiver with vertex set $\{e_1, \ldots, e_n\}$, and $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}_0^n$.

1. A *representation of* Q *over* k *with dimension vector* $\mathbf{d}$ consists of the following:
   - A n-tuple of vector spaces $(V_1, \ldots, V_n)$, where $V_i = k^{d_i}$
   - For each arrow α from e_i to e_j, a linear map $f_\alpha : V_i \to V_j$.

2. Given two representations $\{(V_i)_{1\leq i\leq n}, \{f_\alpha\}\}$, and $\{(W_i)_{1\leq i\leq n}, \{g_\alpha\}\}$, a morphism of representation is a n-tuple of linear maps $\varphi_i : V_i \to W_i$ such that for all arrows α from e_i to e_j, the following diagram commutes:

$$\begin{tikzcd} V_i \arrow[r, "\varphi_i"] \arrow[d, "f_\alpha"'] & W_i \arrow[d, "g_\alpha"] \\ V_j \arrow[r, "\varphi_j"] & W_j. \end{tikzcd}$$

Representations of Q with these morphisms form an abelian category. The monomorphisms, epimorphisms, and isomorphisms end up being what one would expect. For example, a morphism of representations is an isomorphism if and only if all of its constituent linear maps are.

Example 2.13. Q here is the *Kronecker quiver*:

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}$$

For $d, d' \in \mathbb{N}$, a representation of Q with dimension vector (d, d') can then be thought of as a pair of linear maps $f_\alpha, f_\beta : k^d \to k^{d'}$ relative to the following equivalence relation:

$$(f_\alpha, f_\beta) \approx (f'_\alpha, f'_\beta) \Leftrightarrow \exists g \in \mathrm{Aut}_k(K^d), h \in \mathrm{Aut}_k(k^{d'}) \text{ such that } f_\alpha g = hf'_\alpha \text{ and } f_\beta g = hf'_\beta.$$

Classifying the representations of this quiver up to isomorphism is equivalent to classifying the simultaneous normal forms of the linear maps f_α and f_β. This was a problem solved by Kronecker (whence the name, despite the fact that Kronecker's time was long before quivers were developed).

If I is an admissible ideal, we can also define a representation of Q modulo I to be a representation $\{(V_i)_{1\leq i\leq n}, \{f_\alpha\}\}$ of Q such that the f_αs "agree with the relations in I". More precisely, suppose that $\sum_{1\leq j\leq s} a_j p_j \in I$, with $s \in \mathbb{N}$, each $a_j \in k$, and each $p_j =$

$\alpha_{j,l_j} \cdots \alpha_{j,1}$ a path with the $\alpha_{j,m}$s arrows. Then we require that $\sum_{1 \leq j \leq s} a_j f_{\alpha_{j,l_j}} \cdots f_{\alpha j,1} = 0$.

Given a representation $(\{V_i\}_{1 \leq i \leq n}, \{f_\alpha\})$ of Q modulo I, we can construct a left kQ/I-module M in the following way: let M have underlying vector space $\bigoplus_{i=1}^n V_i$. Then, for a vertex e_i, let $e_i(v_1, \ldots, v_n) = (0, \ldots, v_i, \ldots, 0)$, and for an arrow α from e_i to e_j, define $\alpha(v_1, \ldots, v_n) = (0, \ldots, f_\alpha(v_i), \ldots, 0)$ (note that here, the $f_\alpha(v_i)$ will appear in the jth coordinate). Since the f_αs agree with the relations in I, this induces a well-defined left kQ/I-structure on M.

Conversely, if $M \in kQ/I$-mod, we can let $V_i = e_i M$, and, for each arrow α with start(α) and end(α) $= e_j$, define a linear map $f_\alpha : V_i \to V_j$ by

$$f_\alpha : m \mapsto \alpha m.$$

Then the vector spaces $V_1, \ldots, V_n$ together with the f_αs comprise a representation of Q modulo I.

Proposition 2.14. There is an equivalence of abelian categories between representations of Q modulo I and kQ/I-mod. The above constructions describe the restriction of this equivalence to the objects of the categories.

Given the tangential nature of representations to this document, we omit the details of this construction. We will be looking at things from the viewpoint of left kQ/I-modules.

3 A Primer on Parameterizing Varieties of Finite-Dimensional Left Λ-Modules

Throughout this section, let Q be a quiver with vertices $e_1, \ldots, e_n$, and $\Lambda = kQ/I$ a path algebra modulo relations with Jacobson radical $J = \{\alpha + I \mid \alpha \text{ is an arrow}\}$.

3.1 The Affine Parameterizing Varieties

In the following, let A be a basic, finite-dimensional algebra. We realize A as the quotient of a free algebra $k\langle x_1, \ldots, x_r\rangle/R$, where R is some ideal of relations. We begin by defining an affine parameterizing variety for left A-modules with fixed dimension.

Definition 3.1. Take A as above. We define

$$\mathbf{Mod}_d(A) = \left\{ (a_i)_{1\leq i\leq r} \in \prod_{1\leq i\leq r} \mathrm{Mat}_d(k) \mid f(a_1, \ldots, a_r) = 0 \text{ for all } f \in R \right\}.$$

Whenever it will not introduce confusion, we will write the points of $\mathbf{Mod}_d(A)$ in the form (x_i).

That the matrices satisfy the relations in A boils down to polynomial conditions on the entries of those matrices. Hence $\mathbf{Mod}_d(A)$ is closed subvariety of $\prod_{1\leq i\leq r} \mathrm{Mat}_d(k)$ and so is affine. This variety comes with a $\mathrm{Gl}_d(k)$-action by conjugation: for $g \in \mathrm{Gl}_d(k)$, define $g.(a_i) = (ga_ig^{-1})$. We will write $\mathrm{Gl}_d.((x_i))$ for the orbits under this action.

There is a natural map from $\mathbf{Mod}_d(A)$ to the set of isomorphism classes of left A-modules with dimension d which interacts nicely with the orbits of the $\mathrm{Gl}_d(k)$-action. Given $x = (x_i) \in \mathbf{Mod}_d(A)$, we construct a left A-module in the following way: let $M = k^d$ as a vector space. For a generator x_i, and $m \in M$, we set $x_i.m = a_im$. Since the x_is agree with the relations in A, this induces a well-defined left A-module structure on M.

Notation. With $x \in \mathbf{Mod}_d(A)$ as above, we write $M(x)$ for the corresponding left A-module.

Conversely, if M is a left A-module, we have that each map $m \mapsto a_i.m$ is a vector space endomorphism on M. Fixing a basis, this yields a tuple of matrices $(x_i) \in \mathbf{Mod}_d(A)$. This dependence on a choice of basis of M suggests a connection with conjugation by matrices in $\mathrm{Gl}_d(k)$. Indeed, we have the following:

Lemma 3.2. The above constructions induces a bijection

$$\{\text{Orbits in } \mathbf{Mod}_d(A) \text{ of the } \mathrm{Gl}_d(k)\text{-action}\} \longleftrightarrow \{\text{Isomorphism classes of } A\text{-modules}\}$$
$$\mathrm{Gl}_d .(x) \longmapsto [M(x)].$$

□

3.2 Subdivisions of $\mathbf{Mod}_d(A)$

Now that we have constructed $\mathbf{Mod}_d(A)$, we wish to understand it as well as we can. In particular, we are interested in the irreducible components of this variety. We will try to get at them via a series of refinements.

Definition 3.3. Let $M \in A$-mod. The *dimension vector* of M is

$$\mathbf{dim}(M) = (\dim_k(e_1 M), \dim_k(e_2 M), \ldots, \dim_k(e_n M)) \in \mathbb{N}_0^n.$$

Note that if $\mathbf{dim}(M) = (d_1, \ldots, d_n)$, then $\dim_k M = \sum_{1\leq i\leq n} d_i$. Generally, for an dimension vector $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}_0^n$, we define $|\mathbf{d}| = \sum_{1\leq i\leq n} d_i$.

One can think of the dimension vector as describing the number of times each vertex "appears" in the module M. More precisely, the coordinate d_i in $\mathbf{d}$ is the number of times the simple module S_i appears in a composition series for M.

Theorem 3.4 ([8], Corllary 1.4)**.** The connected components of $\mathbf{Mod}_d(A)$ are precisely the subvarieties of the form

$$\{x \in \mathbf{Mod}_d(A) \mid \mathbf{dim}(M(x)) = \mathbf{d}\},$$

where $\mathbf{d}$ traces the dimension vectors with $|\mathbf{d}| = d$.

The irreducible components of of the $\mathbf{Mod}_d(A)$s are contained in the connected components. Since we will be dealing only with path algebras modulo relations in the following, we will pass to affine varieties which encode the Λ-modules with fixed dimension vector $\mathbf{d}$ more efficiently in this case.

Definition 3.5. $\Lambda = kQ/I$ is a path algebra modulo relations and Arr_Q the set of arrows in Q.

$$\mathbf{Mod}_{\mathbf{d}}(\Lambda) = \{(x_\alpha)_{\alpha \in \mathrm{Arr}_Q} \mid x_\alpha \in \mathrm{Hom}_k(k^{d_{\mathrm{start}(\alpha)}}, k^{d_{\mathrm{end}(\alpha)}}) \text{ such that the } x_\alpha \text{ satisfy the relations in } I\}.$$

That a point $(x_\alpha) \in \mathbf{Mod}_{\mathbf{d}}(\Lambda)$ satisfies the relations in I means the following: write $x_p = x_{\alpha_m} \cdots x_{\alpha_1}$ whenever $p = \alpha_m \cdots \alpha_1 \in kQ$ is a path. If we have a finite linear combination of paths $\sum k_i p_i \in I$, then $\sum k_i x_{p_i} = 0$.

Example 3.6. Let Q be the Kronecker quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_2"'] & 2 \end{tikzcd}$$

and $\Lambda = kQ$. We know from Theorem 3.4 that $\mathbf{Mod}_d\Lambda$ has connected components resulting from the dimension vectors $(2,0)$, $(0,2)$, and $(1,1)$. To describe them, we pass to the more convenient variety of Definition 3.5.

It is easy to see that $\mathbf{Mod}_{(2,0)}(\Lambda)$ and $\mathbf{Mod}_{(0,2)}(\Lambda)$ are both singletons, parameterizing S_1^2 and S_2^2, respectively. The third component is slightly more interesting, giving

$$\mathbf{Mod}_{(1,1)} = \left\{ \left(\begin{pmatrix} 0 & 0 \\ a_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ a_2 & 0 \end{pmatrix} \right) \in (\mathrm{Mat}_2(k))^2 \mid a_1, a_2 \in k \right\}.$$

Then $\mathbf{Mod}_{(1,1)}(\Lambda) \cong \mathbb{A}^2$.

This variety $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$ parameterizes the isomorphism classes of left Λ-modules of dimension vector $\mathbf{d}$. We have a map

$$\mathbf{Mod_d}(\Lambda) \ni x \longrightarrow M(x) \in \Lambda\text{-mod},$$

where $M(x) = \bigoplus_{1 \leq i \leq n} V_i$, where $V_i = k^{d_i}$, as a vector space, and has the left Λ-action induced by the following: if $e_i \in \Lambda$ is a vertex, then e_i acts as the projection map onto V_i on $M(x)$. If α is an arrow from i to j, then $\alpha v_i = x_\alpha v_i \in V_j$ whenever $v_i \in V_i$, and $\alpha v_l = 0$ for $v_l \in V_l \neq V_i$. (note the similarity between this construction and that of a representation fo a quiver in Subsection 2.3.)

The variety $\mathbf{Mod_d}(\Lambda)$ also comes with a a morphic action by a linear algebraic group.

Definition 3.7. Let $\mathbf{d} \in \mathbb{N}_0^n$ be a dimension vector. Then

$$\mathrm{Gl}_{\mathbf{d}}(k) = \prod_{1 \leq i \leq n} \mathrm{Gl}_{d_i}(k).$$

This group has an action on $\mathbf{Mod_d}(\Lambda)$ given by

$$(g_i).(x_\alpha) = (g_{\mathrm{end}(\alpha)} x_\alpha g^{-1}_{\mathrm{start}(\alpha)}).$$

We omit the proof of the following proposition, whose proof closely follows that of Lemma 3.2.

Proposition 3.8. The orbit $\mathrm{Gl}_{\mathbf{d}}(k).x$ consists precisely of the points $y \in \mathbf{Mod_d}(\Lambda)$ for which $M(x) \cong M(y)$.

Example 3.9. Q is the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, bend left, "\beta"] & 3 \arrow[l, bend left, "\gamma_1"] \arrow[l, bend left=80, "\gamma_2"] \arrow[r, "\delta"] & 4 \end{tikzcd}$$

Let $\Lambda = kQ/\langle \text{all paths of length } 3 \rangle$. Take $\mathbf{d} = (1,1,1,1)$. In this case, each $\mathrm{Mat}_k(k^{d_i}, k^{d_j})$

consists of 1×1 matrices. A point in $\mathbf{Mod_d}(\Lambda)$ can therefore be written in the form $x = (x_\alpha, x_\beta, x_{\gamma_1}, x_{\gamma_2}, x_\delta) \in \mathbb{A}^5$. Moreover, we have $\mathrm{Gl}_{\mathbf{d}}(k) = (\mathrm{Gl}_1(k))^4 = (\mathbb{A}^1 \setminus \{(0,0)\})^4$. In the following, we explore the irreducible components of $\mathbf{Mod_d}(\Lambda)$.

First, let $M = \Lambda e_1 \oplus \Lambda e_4$. M is represented by the point y such that $y_{\gamma_1} = y_{\gamma_2} = y_\delta = 0$, and $y_\alpha = y_\beta = 1$. For $g = (g_1, g_2, g_3, g_4) \in \mathrm{Gl}_{\mathbf{d}}(k)$, we therefore have

$$g.y = (g_2 g_1^{-1}, g_3 g_2^{-1}, 0, 0, 0).$$

Therefore $\mathrm{Gl}_{\mathbf{d}} .y = \{(a, b, 0, 0, 0) \in \mathbb{A}^5 \mid a, b \neq 0\}$. We define

$$\mathcal{C}_1 = \overline{\mathrm{Gl}_{\mathbf{d}} .y} \cong \mathbb{A}^2,$$

which is irreducible.

Next, we let $M = \Lambda e_1 / J e_1 \oplus \Lambda e_2 / \langle \gamma_1, \gamma_2 \rangle$. This is represented by the point y whose coordinates are given by $y_\alpha = y_{\gamma_1} = y_{\gamma_2} = 0$, and $y_\beta = y_\delta = 1$. Employing a similar computation to the one above, we define

$$\mathcal{C}_2 = \overline{\mathrm{Gl}_{\mathbf{d}} .y} \cong \mathbb{A}^2.$$

Now, let $k = [k_1 : k_2] \in \mathbb{P}^1$, and let

$$M_k = \Lambda e_1 \oplus \Lambda e_3 / C,$$

where $C = \Lambda \beta \alpha + \Lambda(\alpha - (k_1 \gamma_1 + k_2 \gamma_2))$. Then M_k is represented by the point $y_k = (1, 0, k_1, -k_2, 1)$. Then $\overline{\mathrm{Gl}_{\mathbf{d}} .y_k} \cong \mathbb{A}^3$. We then set

$$\mathcal{C}_3 = \overline{\bigcup_{k \in \mathbb{P}^1} \mathrm{Gl}_{\mathbf{d}} .y_k} \cong \mathbb{A}^4.$$

One can verify that any module with dimension vector $\mathbf{d}$ is represented by a point in one of the irreducible subvarieties $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \subseteq \mathbf{Mod}_{\mathbf{d}}(\Lambda)$. These three varieties are therefore the irreducible componenets of $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$.

3.2.1 A Further Subdivision

An further subdivision of $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$ is in terms of tops. Given a left A-module M, recall that its top is $T = M/JM$. To ease our notation, we will henceforth identify isomorphic semisimple left Λ-modules.

Definition 3.10. Fix a semisimple module $T \in A$-mod and $\mathbf{d} \in \mathbb{N}_0^n$. Define

$$\mathbf{Mod}_{\mathbf{d}}^T = \{x \in \mathbf{Mod}_{\mathbf{d}}(\Lambda) \mid M(x)/JM(x) \cong T\}.$$

Observations 3.11. One readily sees that the sets $\mathbf{Mod}_{\mathbf{d}}^T$ are pairwise disjoint, and their union is all of $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$. Since the top is an isomorphism invariant, Proposition 3.8 gives that $\mathbf{Mod}_{\mathbf{d}}^T$ is stable under the $\mathrm{Gl}_{\mathbf{d}}(k)$-action.

3.3 The Projective Parameterizing Varieties and their Subdivisions

We now turn our attention to a family of projective parameterizing varieties of left A-modules with fixed top T defined in terms of the classical Grassmannian. These varieties admit new useful tools for analysis. Throughout this section, we will develop a hierarchy of varieties, each stable under a natural morphic action by $\mathrm{Aut}_\Lambda(P)$, where P is the distinguished projective cover of T:

$$\mathfrak{Grass}_d^T \supseteq \mathfrak{Grass}_{\mathbf{d}}^T \supseteq \mathfrak{Grass}(\mathbb{S}) \supseteq \mathfrak{Grass}(\sigma).$$

Throughout this section, $\Lambda = kQ/I$ will be a path algebra modulo relations with vertex idempotents $e_1, \ldots, e_n$.

Definition 3.12. Let $T = \bigoplus_{1 \le i \le n} (S_i)^{n_i}$, where $n_i \in \mathbb{N}_0$, be a semisimple module and $P = \bigoplus_{1 \le r \le t} \Lambda z_r$ its projective cover. Let $d \in \mathbb{N}$, and $\tilde{d} = \dim_k P - d$. We write $\mathrm{Gr}(\tilde{d}, JP)$ for the classical Grassmannian of $\tilde{d}$-dimensional vector subspaces of JP (see [14, §1.8] for a transparent introduction to that object). We define

$$\mathfrak{Grass}_d^T = \{C \in \mathrm{Gr}(\tilde{d}, JP) \mid C \text{ is a } \Lambda\text{-submodule of } JP\}.$$

Observation 3.13. $\mathfrak{Grass}_d^T$ is nonempty if and only if $t \le d \le \dim_k P$.

To begin, we show the following:

Proposition 3.14. $\mathfrak{Grass}_d^T$ is a closed subvariety of $\mathrm{Gr}(\tilde{d}, JP)$, whence $\mathfrak{Grass}_d^T$ is a projective variety.

Proof. Let $a_1, \ldots, a_s$ be a generating set for Λ. Then

$$\begin{aligned} \mathfrak{Grass}_d^T &= \{C \in \mathrm{Gr}(\tilde{d}, JP) \mid a_i.C \subseteq C \text{ for all } 1 \le i \le s\} \\ &= \bigcap_{1 \le i \le s} \{C \in \mathrm{Gr}(\tilde{d}, JP) \mid a_i.C \subseteq C\}. \end{aligned}$$

It therefore suffices to show that if f is a vector space endomorphism of JP, then $\{C \in \mathrm{Gr}(\tilde{d}, JP) \mid f(C) \subseteq C\}$ is closed.

Let $a \in k$ not an eigenvalue of f, and set $g = a \cdot \mathrm{id}_V - f$. Then g is an automorphism of V. Note that $f(C) \subseteq C$ if and only if $g(C) = C$. Let $\mathcal{G} = \{C \in \mathrm{Gr}(\tilde{d}, JP) \mid g(C) = C\}$. Consider the map

$$\begin{aligned} \hat{g} : \mathrm{Gr}(\tilde{d}, JP) &\to \mathrm{Gr}(\tilde{d}, JP) \times \mathrm{Gr}(\tilde{d}, JP) \\ C &\mapsto (C, g(C)). \end{aligned}$$

$\hat{g}$ is a morphism of varieties (this follows from the fact that the $\mathrm{Gl}(JP)$-action on $\mathrm{Gr}(\tilde{d}, JP)$

is morphic). Then $\mathcal{G} = \hat{g}^{-1}(\Delta)$, where

$$\Delta = \{(D, D) \mid D \in \mathrm{Gr}(\tilde{d}, JP\},$$

the diagonal in $\mathrm{Gr}(\tilde{d}, JP) \times \mathrm{Gr}(\tilde{d}, JP)$. Δ is a closed subset, and hence $\mathcal{G}$ is closed, as desired. $\square$

The variety $\mathfrak{Grass}_d^T$ comes with a surjection

$$\psi : \mathfrak{Grass}_d^T \to \{\text{isomorphism classes of } d\text{-dimensional left } \Lambda\text{-modules with top } T\}$$
$$C \mapsto [P/C].$$

Indeed, every module M with top T admits P as its distinguished projective cover, and hence is isomorphic to P/C for some submodule C of JP. We also have a morphic action by $\mathrm{Aut}_\Lambda(P)$ given by

$$\varphi.C = \varphi(C).$$

Observation 3.15. The orbits in $\mathfrak{Grass}_d^T$ of the $\mathrm{Aut}_\Lambda(P)$-action coincide with the fibres of the surjection ψ.

Proof. Suppose that $\mathrm{Aut}_\Lambda(P).C = \mathrm{Aut}_\Lambda(P).D$. Then $\varphi(C) = D$ for some $\varphi \in \mathrm{Aut}_\Lambda(P)$. Then φ induces an isomorphism $P/C \to P/D$. Conversely, suppose we have an isomorphism $f : P/C \to P/D$. Let q_C and q_D be the quotient maps of P onto P/C and P/D, respectively. Then the projectivity of P allows us to lift $f \circ q_C$ along q_D to a map $\varphi : P \to P$, making the following diagram commute:

$$\begin{tikzcd} P \arrow[r, dashed, "\varphi"] \arrow[d, "q_C"] & P \arrow[d, "q_D"] \\ P/C \arrow[r, "f"] & P/D. \end{tikzcd}$$

One can check that φ is an isomorphism with $\varphi(C) = D$, as desired. $\square$

Example 3.16 (cf. Example 3.6)**.** Let Q be the Kronecker quiver and $\Lambda = kQ$. By the second part of Observations 3.11, the possible tops of modules of dimension 2 are $S_1^2, S_2^2, S_1 \oplus S_2$, and S_1. Only in the case of S_1 do we have more than one isomorphism class of left Λ-modules, so we let $P = \Lambda e_1$. Λe_1 has basis $\{e_1, \alpha_1, \alpha_2\}$, whence it has dimension 3. Then, in our case, $\tilde{d} = 1$, and so

$$\mathfrak{Grass}_2^{S_1} = \{C \in \mathrm{Gr}(1, \Lambda e_1) \mid C \text{ is a left } \Lambda \text{ submodule}\}.$$

Since $C \subseteq \Lambda e_1$ is a subspace of dimension 1, it is the span of some nonzero vector v, which we may write $ae_1 + b\alpha_1 + c\alpha_2$ for some $a, b, c \in k$. Since kv is a submodule, then, in particular, $\alpha_1 v = \beta v$ for some scalar β. Since $\alpha_1 e_1 = 0$, this implies that $a = 0$, whence $v = b\alpha_1 + c\alpha_2$. It is straightforward to check that the span of any vector of this form is a submodule. We therefore have an isomorphism

$$\mathbf{P}^1 \to \mathfrak{Grass}_2^{S_1}$$
$$[y_1 : y_2] \to k(y_1\alpha_1 + y_2\alpha_2).$$

Note that this is a smaller variety than its affine cousin in Example 3.6. The difference is even more pronounced in larger examples.

On the whole, the projective varieties $\mathfrak{Grass}_d^T$ look rather different from their quasi-affine relatives, $\mathbf{Mod}_d^T$. However, we have a result which allows us to compare the geometry of the orbits of these two varieties under their respective group actions.

Proposition 3.17 (See [3], Proposition 2.2)**.** The map

$$\{\mathrm{Aut}_\Lambda(P) \text{ orbits in } \mathfrak{Grass}_d^T\} \to \{\mathrm{Gl}_d(k) \text{ orbits in } \mathbf{Mod}_d^T\}$$
$$\mathrm{Aut}_\Lambda(P).C \mapsto \mathrm{Gl}_d .x,$$

where C and x represent the same module, induces an inclusion-preserving bijection

$$\{\mathrm{Aut}_\Lambda(P)\text{-stable subsets of } \mathfrak{Grass}_d^T\} \to \{\mathrm{Gl}_d(k)\text{-stable subsets of } \mathbf{Mod}_d^T\}$$

preserving openness, closures, connectedness, irreducibility, and types of singularities.

This allows one to move results back and forth between not only $\mathbf{Mod}_d^T$ and $\mathfrak{Grass}_d^T$, but between each of their $\mathrm{Gl_d(k)}$- and $\mathrm{Aut}_A(P)$-stable subdivisions to follow.

3.3.1 A Further Partition of the varieties $\mathbf{Mod}_d^T$ and $\mathfrak{Grass}_d^T$

Up to this point, we have seen the varieties $\mathbf{Mod}_d^T$ and $\mathfrak{Grass}_d^T$, both parameterizing d-dimensional left Λ-modules with fixed top. Now we cut down further, to modules sharing not only a top, but their entire radical layering. Each of these subvarieties will then admit a particularly convenient open affine cover given by skeletons (see the introduction, definition to come).

Definition 3.18. 1. A *semisimple sequence of dimension d with top T* is a tuple $\mathbb{S} = (\mathbb{S}_0, \ldots, \mathbb{S}_L)$ of semisimple left Λ-modules with $\mathbb{S}_0 = T$ and $\sum_{1\le i\le L} \dim_k \mathbb{S}_i = d$.

2. For $M \in \Lambda$-mod, we write $\mathbb{S}(M)$ for the semisimple sequence

$$(M/JM, JM/J^2M/, \ldots, J^LM/J^{L+1}M),$$

where L is the largest natural number with $J^L \neq 0$.

In the interest of simplifying notation, we identify isomorphic semisimple modules (reducing the first condition above to $\mathbb{S}_0 = T$). In a basic algebra, we are well acquainted with the finitely-many simple modules, namely the Ae_i/Je_is, and so a semisimple sequence is fully described by the discrete invariants giving the multiplicity of each of the simple summands.

Definition 3.19. Let $\mathbb{S}$ be a semismiple sequence of dimension d with top T. Then

$$\mathfrak{Grass}(\mathbb{S}) = \{C \in \mathfrak{Grass}_d^T \mid \mathbb{S}(P/C) = \mathbb{S}\}.$$

The varieties $\mathfrak{Grass}(\mathbb{S})$ form a pairwise-disjoint subdivision of $\mathfrak{Grass}_d^T$ into locally closed subvarieties ([11], Observation 2.11). It is these $\mathfrak{Grass}(\mathbb{S})$s we will be trying to get at, by way of a useful open cover.

3.3.2 Restricting to the Graded Setting

As stated in Observations 2.3 (5), a path algebra kQ carries a grading by path length. If I is a homogeneous admissible ideal, this grading is inherited by the path algebra modulo relations $\Lambda = kQ/I$. Then any distinguished projective cover $P = \bigoplus_{1 \leq r \leq t} \Lambda z_r$ is graded. This leads to $M = P/C$ being graded if and only if C is a homogeneous submodule of P. At this point, we restrict our attention to graded modules M, cutting down our parameterizing varieties in the following way:

Definition 3.20.

$$\text{Gr-}\,\mathfrak{Grass}_{\mathbf{d}}^T = \{C \in \mathfrak{Grass}_{\mathbf{d}}^T \mid C \text{ is a homogeneous submodule of } P\}.$$

We further define

$$\text{Gr-}\,\mathfrak{Grass}(\mathbb{S}) = \mathfrak{Grass}(\mathbb{S}) \cap \text{Gr-}\,\mathfrak{Grass}_{\mathbf{d}}^T.$$

These varieties then parameterize *graded* left Λ-modules, again by the map $C \mapsto P/C$.

Observation 3.21. In the graded setting, we are afforded the convenience of the radical layering coinciding with the grading. Recall that, in our setting, the Jacobson radical J of Λ is the ideal generated by the residue classes of the arrows in kQ. If $\Lambda = kQ/I$ with I homogeneous, it follows that J^l is the ideal generated by the residue classes of the paths of length l in kQ. If $M \in \Lambda$-mod is graded, we have an isomorphism of vector spaces between the l^{th} homogeneous part of M and $J^l M / J^{l+1} M$.

4 A Useful Open Cover of Gr-$\mathfrak{Grass}(\mathbb{S})$

4.1 Cutting Down to the Monomial Case

As before, Q is a quiver with vertices $e_1, \ldots, e_n$, and $\Lambda = kQ/I$ is a path algebra modulo relations with Jacobson radical J. We let $M \in \Lambda$-mod with distinguished projective cover $P = \bigoplus_{1 \leq r \leq t} \Lambda z_r$.

Definitions 4.1. If I is generated by paths, we say that Λ is a *monomial* path algebra. If $I = J^L$ for some natural number L, we say that it is *truncated*.

Note that if Λ is truncated, then, in particular, it is a monomial algebra.

In the following, we will define the skeleton of M. One may define skeleta for general path algebras modulo relations. However, we can save ourselves an enormous load of technical baggage by restricting our definition to the monomial case. The central theorem of this thesis concerns only truncated path algebras, so this narrowing of our focus costs us nothing for our own purposes. Note that it is possible, (indeed, very useful) to define skeleta in the general case. Such a construction can be found in [11]

We now broaden our notion of a path a little bit.

Definition 4.2. Given a distinguished top element z_r, there is an injection $\Lambda e(r) \hookrightarrow \Lambda z_r$ given by $pe(r) \mapsto pz_r$. A *path starting in* z_r is the image of a path starting in $e(r)$ by this injection. If $p = p_2 p_1$, we call p_1 an *initial subpath*. With this injection, the grading in $\Lambda e(r)$ gives a sensible definition for the length of p, which is just its length as a path when included in $\Lambda e(r)$. For a vertex e_i, we say that p *ends in* e_i if $e_i p = p$.

Note that this notion of a path is well-defined only in the monomial case. In the general case, one risks the chosen paths suffering from identity crises. For example, if two paths $pe_i, qe_i \in kQ$ are identified in kQ/I, then the notion of intial subpaths of pe_i and qe_i makes sense only in kQ, but not in kQ/I. To deal with this, in the general case, skeleta are defined to be bases for a left kQ-module which is a cousin of P, rather than P itself.

We offer a warning: the projective module P might have "multiple copies" of some Λe_i if there are some r and s with $e(r) = e(s)$. While a path starting in z_r might correspond to the same "physical" path in kQ as some path starting in z_s, they still represent distinct paths in P, and it is vital for our purposes that we treat them so.

Definition 4.3. Let $q : P \twoheadrightarrow M$ be the surjection partnered with the distinguished projective cover P. Let $\sigma \subseteq P$ consist of paths in P. Write σ_l for the set of paths of length l in σ. We call σ a *skeleton for M* if it satisfies

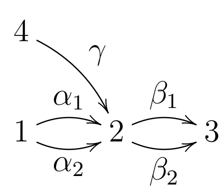
- For each natural number l, $q(\sigma_l)$ induces a basis for $J^l M / J^{l+1} M$.
- σ is closed under initial subpaths. That is, if $p_2 p_1 \in \sigma$, then $p_1 \in \sigma$.

What we've defined is a special vector space basis for M (this follows from the first bullet point). In particular, σ is finite (indeed, $|\sigma| = \dim_k M$).

Observation 4.4. From Observation 3.21, the first bullet point is equivalent to requiring that $q(\sigma_l)$ be a basis for M_l, the l^{th} graded piece of M.

One can visualize a skeleton as a forest of rooted trees, where each tree "hangs" from a distinguished top element z_r.

Example 4.5. Let $\Lambda = kQ$ where Q is the quiver



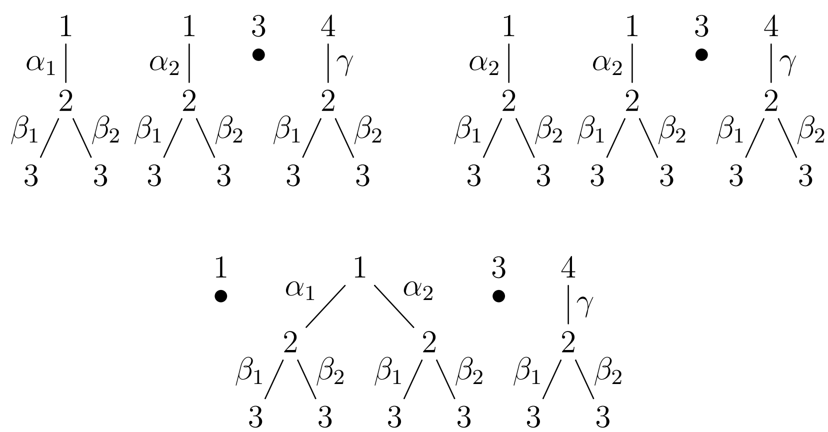
Let $T = S_1^2 \oplus S_3 \oplus S_4$. T has distinguished projective cover $P = \bigoplus_{1 \le r \le 4} \Lambda z_r$, with $e(1) = e(2) = e_1, e(3) = e_3$, and $e(4) = e_4$. Consider the left module $M = P/C$, where C is generated by $\alpha_1 z_1 - \alpha_2 z_1$ and $\alpha_1 z_1 - \alpha_1 z_2$. Then the following are skeletons for M:

$$\{z_1, \alpha_1 z_1, \beta_1\alpha_1 z_1, \beta_2\alpha_1 z_1\} \sqcup \{z_2, \alpha_2 z_2, \beta_1\alpha_2 z_2, \beta_2\alpha_2 z_2\} \sqcup \{z_3\} \sqcup \{z_4, \gamma z_4, \beta_1\gamma z_4, \beta_2\gamma z_4\}$$

$$\{z_1, \alpha_2 z_1, \beta_1\alpha_2 z_1, \beta_1\alpha_2 z_1\} \sqcup \{z_2, \alpha_2 z_2, \beta_1\alpha_2 z_2, \beta_2\alpha_2 z_2\} \sqcup \{z_3\} \sqcup \{z_4, \gamma z_4, \beta_1\gamma z_4, \beta_2\gamma z_4\}$$

$$\{z_1\} \sqcup \{z_2, \alpha_1 z_2, \alpha_2 z_2, \beta_1\alpha_1 z_2, \beta_2\alpha_1 z_2, \beta_1\alpha_2 z_2, \beta_2\alpha_2 z_2\} \sqcup \{z_3\} \sqcup \{z_4, \gamma z_4, \gamma\beta_1 z_4, \beta_2\gamma z_4\}.$$

We may visualize these skeletons as forests of rooted trees. We label each vertex by the index of the corresponding idempotent in kQ:



Since the paths of length l form a basis for $J^l M/J^{l+1}M$, one can "read" the radical layering off of the rows of each of these diagrams. In this example, we have

$$M/JM = S_1^2 \oplus S_3 \oplus S_4$$
$$JM/J^2M = S_2^3$$
$$J^2M/J^3M = S_3^6,$$

and $J^3 = 0$.

Proposition 4.6. Every module $M \in \Lambda$-mod with distinguished projective cover P has a skeleton.

Proof. We construct σ_l by induction on l. Set $\sigma_0 = \{z_1, \ldots, z_t\}$. Then suppose we have σ', a skeleton for M/J^lM. If $J^{l+1}M = 0$, then set $\sigma = \sigma'$, and we are done. So suppose that $J^{l+1}M \neq 0$. Then, by appending arrows to paths in σ'_l, one constructs a set of paths σ_{l+1} inducing a basis for $J^{l+1}M/J^{l+2}M$ such that every element has an initial subpath in σ'_l. Then take $\sigma = \sigma' \cup \sigma_{l+1}$. $\square$

4.2 Abstract Skeletons and the Radical Layering

A key part of the definition of a skeleton is its relationship with the radical layering. Given our varieties $\mathfrak{Grass}(\mathbb{S})$ parameterizing modules with a particular radical layering, we'd like to carve them up further by skeletons. To do this, we divorce the notion of skeleton from any particular Λ-module.

Definitions 4.7.

1. Let $P = \bigoplus_{i \leq r \leq s} \Lambda z_r$, with distinguished top elements $z_1, \ldots, z_s$, and $T = P/JP$. An *abstract skeleton with top T* is a nonempty finite set of paths in P which is closed under right subpaths.

2. Given a semisimple sequence $\mathbb{S} = (\mathbb{S}_0, \mathbb{S}_1, \ldots, \mathbb{S}_L)$, We say that an abstract skeleton σ is *compatible with* $\mathbb{S}$ if, for each natural number $l \leq L$ and each simple left Λ-module S_i, the multiplicity of S_i in $\mathbb{S}_l$ is equal to the number of paths of length l ending in the vertex i in σ.

If an abstract skeleton σ happens to satisfy Definition 4.3 for a particular module M, then we can go right on saying that σ is a skeleton for M. However, the definition of an abstract skeleton doesn't give any guarantee that there will exist such a module.

We now describe a system of affine coordinates that can be imposed on Gr-$\mathfrak{Grass}(\sigma)$. The strategy is to employ paths that are "almost" in σ. That is, paths in σ with one arrow added to the end. These paths can then be broken down into a linear combination of elements of the skeleton. This process then yields an algorithm which we can apply to paths in general.

As before, σ is a skeleton with top T, and T has distinguished projective cover $P = \bigoplus_{i \leq r \leq s} \Lambda z_r$.

Definition 4.8. Let $T \in \Lambda$-mod be semisimple, d a natural number, and σ an abstract skeleton with top T. We define

$$\text{Gr-}\mathfrak{Grass}(\sigma) = \{C \in \text{Gr-}\mathfrak{Grass}_d^T \mid \sigma \text{ is a skeleton of } P/C\}.$$

Suppose we have some $C \in \text{Gr-}\mathfrak{Grass}(\sigma)$. Let σ_+ denote the set of paths in σ of positive length. Then $\sigma_+ \subseteq JP$. Since σ is, in particular, a basis for P/C, it follows that $C \cap \operatorname{Span}(\sigma_+) = \{0\}$ (here, "Span" simply denotes the linear span in the k-vector space JP). Putting this together, we have that $JP = C \oplus \operatorname{Span}(\sigma_+)$. We define

$$\operatorname{Schu}(\sigma) = \{C \in \operatorname{Gr}(\dim P - d, JP) \mid JP = C \oplus \operatorname{Span}(\sigma_+)\}.$$

$\operatorname{Schu}(\sigma)$ is an open Schubert cell in the classical Grassmannian ([9, p. 195]. From the above, $C \in \operatorname{Schu}(\sigma)$ if and only if σ is a vector space basis for P/C. If $\text{Gr-}\mathfrak{Grass}(\sigma) \neq \phi$, then there exists a unique semisimple sequence $\mathbb{S}$ such that σ is compatible with $\mathbb{S}$. It is then straightforward to check that σ is a skeleton for P/C if and only if $C \in \operatorname{Schu}(\sigma) \cap \text{Gr-}\mathfrak{Grass}(\mathbb{S})$. We therefore have

Observation 4.9. Let σ be a skeleton with top T and $\mathbb{S}$ a semisimple sequence also with top T. Then $\text{Gr-}\mathfrak{Grass}(\sigma)$ is open in $\text{Gr-}\mathfrak{Grass}(\mathbb{S})$.

In general, $\text{Gr-}\mathfrak{Grass}(\sigma)$ fails to be open in $\text{Gr-}\mathfrak{Grass}_d^T$. Unlike with $\text{Gr-}\mathfrak{Grass}(\mathbb{S})$, we do not have that $\text{Gr-}\mathfrak{Grass}(\sigma)$ is equal to $\text{Gr-}\mathfrak{Grass}_d^T \cap \operatorname{Schu}(\sigma)$. This is because $\text{Gr-}\mathfrak{Grass}_d^T \cap \operatorname{Schu}(\sigma)$ may contain points representing modules M which have σ as a plain vector space basis, but for which σ fails to be a skeleton for M (that is, the sets σ_l fail to induce bases for each radical layer).

Having established that Gr- $\mathfrak{Grass}(\sigma)$ is open in Gr- $\mathfrak{Grass}(\mathbb{S})$, we now describe a system of affine coordinates for Gr- $\mathfrak{Grass}(\sigma)$.

Definition 4.10. A nonzero path $q \in P$ is called *σ-critical* if $q \notin \sigma$, but every proper initial subpath of q is in σ. That is, $qz_r = \alpha p z_r$, where α is an arrow and $pz_r \in \sigma$.

Suppose that σ is a skeleton for a module $M = P/C$ and that $\alpha p z_r$ is a σ-critical path of length l. Since the set σ_l of paths of length l in σ induces a basis for M_l (see Observation 3.21), we have

$$\alpha p z_r + C = \left(\sum_{qz_r \in \sigma_l} a_q q z_r \right) + C$$

for *unique* scalars $a_q \in k$. In fact, we may go further. We have $\mathrm{end}(\alpha)\alpha p z_r = \alpha p z_r$, but $\mathrm{end}(\alpha) q = 0$ if $\mathrm{end}(\alpha) \neq \mathrm{end}(q)$. Therefore, all paths with different end points from α "disappear" from the linear combination. All paths with different starting points from $e(r)$ similarly vanish. In light of this, we define the following notation:

Notation. We write $\sigma(\alpha p z_r)$ for the set of all paths in σ with the same length, starting vertex, and ending vertex as $\alpha p z_r$. The above then becomes

$$\alpha p z_r + C = \left(\sum_{q \in \sigma(\alpha p z_r)} a_q q z_r \right) + C.$$

We will show that the unique scalars a_q appearing above yield affine coordinates for Gr- $\mathfrak{Grass}(\sigma)$. To that end, we lump the sets $\sigma(\alpha p z_r)$ together into the disjoint union

$$S = \bigsqcup_{\alpha p z_r \ \sigma\text{-critical}} \{\alpha p z_r\} \times \sigma(\alpha p z_r).$$

The set S gives us the coordinates for an affine space parameterizing Gr- $\mathfrak{Grass}(\sigma)$.

Lemma 4.11. Let $P = \bigoplus_{1 \leq r \leq t} \Lambda z_r$, and $\sigma \subseteq P$ be an abstract skeleton. Take S to be the set described above. Then there exists an isomorphism of varieties

$$\psi : \mathbb{A}^{|S|} \longrightarrow \text{Gr-}\,\mathfrak{Grass}(\sigma).$$

Proof. Without loss of generality, we may assume that $P = \Lambda e_i$ for some vertex e_i. We index the coordinates of $\mathbb{A}^{|S|}$ by the set S above. The discussion following Definition 4.10 gives us a map Gr- $\mathfrak{Grass}(\sigma) \to \mathbb{A}^{|S|}$. We therefore turn our attention to the reverse direction. Let $Y \in \mathbb{A}^{|S|}$. We now define $C \subseteq P$ to be the submodule generated by linear combinations of the form

$$\alpha p e_i - \sum_{q \in \sigma(\alpha p e_i)} a_q q e_i, \tag{1}$$

where $\alpha p e_i$ is σ critical and $a_q \in k$ is the affine coordinate of Y indexed by $(\alpha p e_i, q) \in \{\alpha p e_i\} \times \sigma(\alpha p e_i) \subseteq S$. Note that C is homogeneous, for we have just written down a set of homogeneous generators. For a proof that this assignment and its inverse are morphisms, see [4] for an analagous setting.

We wish to show that σ is a skeleton for P/C. To do this, we first prove by induction on l that if pe_i is a path of length l, then there exist scalars b_q such that

$$pe_i + C = \left(\sum_{qe_i \in \sigma_l} b_q q e_i \right) + C.$$

Suppose that this holds for all paths of length $l-1$. Write $pe_i = \alpha p' e_i$, where α is an arrow and p' a path of length $l-1$. Then there exist scalars $b_{q'}$ such that

$$\alpha p' e_i + C = \alpha \left(\sum_{q' e_i \in \sigma_{l-1}} b_{q'} q' e_i \right) + C.$$

It now suffices to show that each $\alpha q' e_i$ is a linear combination of paths in σ_l. If $\alpha q' e_i = 0$, we have nothing to do. If $\alpha q' e_i \in \sigma$, we still have nothing to do. The remaining possibility is that $\alpha q' e_i$ is σ-critical, in which case the relations (1) of C give

$$\alpha q' e_i + C = \left(\sum_{q \in \sigma(\alpha q' e_i)} a_q q e_i \right) + C,$$

a linear combination of paths in σ_l. For linear independence of σ in P/C, see [11] Lemma 3.13.

It is straightforward to check that the correspondences given are inverse bijections. Indeed, the relations picked in (1) are precisely what one gets out in the discussion following Definition 4.10. $\square$

Put together, Observation 4.9 and Lemma 4.11, and Proposition 4.6, yield

Theorem 4.12. $\{\text{Gr-}\mathfrak{Grass}(\sigma)\}_\sigma$ is an open affine cover of $\text{Gr-}\mathfrak{Grass}(\mathbb{S})$, where σ runs through the abstract skeleta compatible with $\mathbb{S}$.

5 Proof of the Main Theorem

We are now ready to prove the main theorem of the thesis, which was first stated in general and proved for the ungraded case in [3], Theorem 5.3. As before, $\Lambda = kQ/I$ is a path algebra modulo relations with Jacobson radical J, $P = \bigoplus_{1 \le r \le t} \Lambda z_r$ is a projective module with a full sequence of distinguished top elements $z_1, \dots, z_t$.

Notation. For a skeleton $\sigma \subseteq P$, we write σ_{li} for the set of paths of length l in σ ending in e_i.

Definition 5.1. The *length* of the algebra Λ is the largest natural number L such that $J^L \neq 0$. (This is simply one less than the Loewy length of Λ.)

First, we will need a geometric lemma.

Lemma 5.2. Let V be a (classical, noetherian) variety and $(U_i)_{i \in I}$ an open cover such that

- U_i is irreducible, rational, and smooth for each $i \in I$.

- $U_i \cap U_j \neq \phi$ for all $i, j \in I$.

Then V is irreducible, rational, and smooth.

Proof. We first show that V is irreducible. Let $U \subseteq V$ be a nonempty open subset. If $U \cap U_i \neq \phi$ for each $i \in I$, then the irreducibility of the U_is will give that $U \cap U_i$ is dense open in each U_i and therefore in their union, V. Suppose for a contradiction that $U \cap U_i = \phi$ for some $i \in I$. Evidently, there exists $j \in J$ for which $U \cap U_j$ is nonempty and therefore dense. By hypothesis, $U_i \cap U_j \neq \phi$, and therefore $U_i \cap U_j$ is dense open in U_j. But then we have two disjoint dense open subsets of U_j, which cannot happen. We therefore have a contradiction, and $U \cap U_i \neq \phi$ for all $i \in I$. Each U_i is then a dense open rational subset, whence V is rational. Smoothness is immediate from the existence of an open cover by smooth subvarieties. $\square$

Theorem 5.3. [cf. [3], Theorem 5.3] Let Q be a quiver and Λ a truncated path algebra. If $\mathbb{S}$ is a semisimple sequence such that Gr-$\mathfrak{Grass}(\mathbb{S})$ is nonempty, then Gr-$\mathfrak{Grass}(\mathbb{S})$ is an irreducible, rational, smooth variety. More specifically, if σ is a skeleton compatible with $\mathbb{S}$, then Gr-$\mathfrak{Grass}(\sigma)$ is a dense open subset of Gr-$\mathfrak{Grass}(\mathbb{S})$, and Gr-$\mathfrak{Grass}(\sigma) \cong \mathbb{A}^N$ for some natural number N dependent only on $\mathbb{S}$.

Proof. The last assertion is simply Lemma 4.11, where $N = |S|$ for the set S described there. Our strategy is to show that the open cover

$$\{\text{Gr-}\mathfrak{Grass}(\sigma)\}_{\sigma \text{ compatible with } \mathbb{S}}$$

of Gr-$\mathfrak{Grass}(\mathbb{S})$ satisfy the conditions of Lemma 5.2. To do this, we will, for any two skeletons σ and $\tilde{\sigma}$ compatible with $\mathbb{S}$, construct a homogeneous submodule C of P such that both σ and $\tilde{\sigma}$ are distinguished skeletons of P/C. This will yield $C \in$ Gr-$\mathfrak{Grass}(\sigma) \cap$ Gr-$\mathfrak{Grass}(\tilde{\sigma})$. The rest of the hypothesis come from the fact that Gr-$\mathfrak{Grass}(\sigma) \cong \mathbb{A}^N$.

We will construct C by induction on the length L of Λ, together with a bijection $f : \tilde{\sigma} \to \sigma$ satisfying

- For each $l \leq L-1$ and $i \leq n$, $f(\tilde{\sigma}_{li}) = \sigma_{li}$.
- $\tilde{p} - f(\tilde{p})$ is a homogeneous element of C lying in $C_{\text{length}(\tilde{p})}$.
- f acts as the identity on $\sigma \cap \tilde{\sigma}$.
- If $pz_r \notin \sigma \cup \tilde{\sigma}$, then $pz_r \in C$.

The first conditions tells us that f preserves path length and ending vertex. The second yields that, furthermore, the sets of residue classes $(\tilde{p} + C)$ and $(f(\tilde{p}) + C)$ yield the same vector space basis for P/C. The third will prove useful in our induction.

Let us begin our induction. Suppose $L = 0$. Then $\Lambda = \Lambda/J$, whence it is semisimple. Then Gr- $\mathfrak{Grass}_d^T \neq \phi$ if and only if $d = t$ (see Observation 3.13). In case $d = t$, Gr- $\mathfrak{Grass}_d^T$ consists of a single point representing the module T, and the result is trivial.

Now suppose $L > 1$, and that the result holds for all algebras of length at most $L - 1$. Then it kicks in for the algebra Λ/J^{L-1}, our path algebra truncated one step shorter. This gives us a homogeneous submodule C' of the "clipped" projective cover $P/J^{L-1}P$ compatible with σ' and $\tilde{\sigma}'$, where σ' and $\tilde{\sigma}'$ are the clipped skeletons we get by deleting all paths of length L from σ and $\tilde{\sigma}$, and a bijection $f' : \tilde{\sigma}' \to \sigma'$ satisfying the above conditions.

σ and $\tilde{\sigma}$ are both compatible with $\mathbb{S}$, so, in particular, $|\tilde{\sigma}_{Li}| = |\sigma_{Li}|$ for each i. Note that σ_L is the disjoint union of all the σ_{Li}s. We may therefore extend f' to a bijection $f : \tilde{\sigma} \to \sigma$ such that f fixes $\sigma \cap \tilde{\sigma}$, and $f(\tilde{\sigma}_{Li}) = \sigma_{Li}$ for each i. Having built f, we will use it to construct a module $C = \Lambda C' + M$, with M chosen judiciously (note: by $\Lambda C'$, we mean the submodule generated by C', a priori a subset of P', considered as a subset of P). We will need

1. If $pz_r \notin \sigma \cup \tilde{\sigma}$, then $pz_r \in C$.

2. If $\tilde{p} \in \tilde{\sigma}$, then $f(\tilde{p}) - \tilde{p} \in C_{\text{length}(\tilde{p})}$.

3. C is a homogeneous submodule of P, and both σ and $\tilde{\sigma}$ are skeletons for P/C.

The first condition comes almost for free (in that it doesn't merit any additions to the module M). We will then go "piece-by-piece" through the skeletons to construct M so that the second condition is satisfied. The third condition will follow from this construction.

The inductive hypothesis yields conditions 1 and 2 for all paths of length at most $L-1$, so we will turn our attention to paths of length L. Henceforth, $p \in \Lambda$ will be a path of length $L-1$ and $\alpha \in \Lambda$ an arrow. First, if $pz_r \notin \sigma \cup \tilde{\sigma}$, then our inductive hypothesis ensures that $pz_r \in C'$. Then $\alpha p z_r \in \Lambda C' \subseteq C$, yielding condition 1.

The meat of the construction deals with condition 2. For this, suppose that $pz_r \in \sigma \cup \tilde{\sigma}$. For all cases, we will construct a homogeneous submodule $C(\alpha p z_r) \subseteq P_L$, and then set

$$M = \sum C(\alpha p z_r).$$

<u>Case 1</u>: Suppose that $\alpha p z_r \notin \sigma \cup \tilde{\sigma}$. Then set $C(\alpha p z_r) = k(\alpha p z_r)$.

<u>Case 2</u>: Suppose $pz_r \in \sigma \cap \tilde{\sigma}$.

- If $\alpha p z_r \in \sigma \cap \tilde{\sigma}$, as well, then $f(\alpha p z_r) = \alpha p z_r$, and we needn't add anything to satisfy condition 2; set $C(\alpha p z_r) = 0$.

- If $\alpha p z_r \in \sigma \setminus \tilde{\sigma}$, then set $C(\alpha p z_r) = k(\alpha p z_r - f^{-1}(\alpha p z_r))$. Similarly, if $\alpha p z_r \in \tilde{\sigma} \setminus \sigma$, then set $C(\alpha p z_r) = k(\alpha p z_r - f(\alpha p z_r))$.

<u>Case 3</u>: Suppose that $pz_r \in \sigma \setminus \tilde{\sigma}$. Then, since skeletons are closed under right subpaths, $\alpha p z_r \notin \tilde{\sigma}$, and $\alpha p z_r \in \sigma$ (for $\alpha p z_r \in \sigma \cup \tilde{\sigma}$). Then set $C(\alpha p z_r) = k(\alpha p z_r - f^{-1}(\alpha p z_r))$.

The final case, $pz_r \in \tilde{\sigma} \setminus \sigma$, is symmetric to case 2. Setting M equal to the sum of these terms as above ensures that $C = \Lambda C' + M$ satisfies properties 1 and 2.

It remains to check that the third property holds. First, it is not hard to see that C is homogeneous: C', and therefore $\Lambda C'$, is homogeneous by the inductive hypothesis.

Moreover, each nonzero $C(\alpha p z_r)$ is generated by a linear combination of paths of length L and is therefore homogeneous. It follows that C is homogeneous.

Finally, we need for σ and $\tilde{\sigma}$ to be skeletons for P/C. Our induction hypothesis has already given us that σ_l and $\tilde{\sigma}_l$ induce bases for $(P/C)_l$ for $1 \leq l \leq L-1$. We need to show that σ_L induces a basis for $(P/C)_L = P_L/C_L$. Indeed, by condition 2 above, we only need to show this for σ. It suffices to show that $P_L = C_L \oplus \mathrm{Span}(\sigma_L)$ as k-vector spaces. It is clear from the construction that $C_L \cap \mathrm{Span}(\sigma_L) = \{0\}$. Once again, let p be a path of length $L-1$ and α an arrow. σ_{L-1} induces a basis for $(P/C)_{L-1}$, so there exist $u, v \in k$, $c \in C$, and $h \in \mathrm{Span}(\sigma_{L-1})$ such that $p = uc + vh$. Then $\alpha p = u\alpha c + v\alpha h$. Since C is a submodule, $u\alpha c \in C$. αh may be written as a linear combination

$$\alpha h = \sum_{q \in \sigma_{L-1}} a_q \alpha q.$$

We define (note the differing indices of summation)

$$c' = \sum_{\substack{q \in \sigma_{L-1} \\ \alpha q \notin \sigma_L}} a_q \alpha q, \quad h' = \sum_{\substack{q \in \sigma_{L-1} \\ \alpha q \in \sigma_L}} a_q \alpha q.$$

Then $c' \in C$ (see Case 1 above), $h' \in \mathrm{Span}(\sigma)$, and

$$\alpha p = (u\alpha c + vc') + vh',$$

with $u\alpha c + vc' \in C_L$ and $vh' \in \mathrm{Span}(\sigma_L)$, whence $P_L = C_L \oplus \mathrm{Span}(\sigma_L)$, as desired. $\square$

Beyond this theorem, we offer the following conjecture:

Conjecture. Let $\Lambda = kQ/I$ a path algebra modulo relations, where I is generated by paths. If $\mathbb{S}$ is a semisimple sequence such that Gr- $\mathfrak{Grass}(\mathbb{S})$ is nonempty, then the irreducible componenets of Gr- $\mathfrak{Grass}(\mathbb{S})$ are irreducible, rational and smooth.

References

[1] Frank Anderson and Kent Fuller. *Rings and categories of modules*. Springer-Verlag, New York, 1992.

[2] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras*. Cambridge University Press, Cambridge New York, 2006.

[3] E. Babson, B. Huisgen-Zimmermann, and R. Thomas. Generic representation theory of quivers with relations. *Journal of Algebra*, 322(6):1877 – 1918, 2009.

[4] Klaus Bongartz and Birge Huisgen-Zimmermann. The geometry of uniserial representations of algebras ii. alternate viewpoints and uniqueness. *Journal of Pure and Applied Algebra*, 157(1):23 – 32, 2001.

[5] Ju.A. Drozd. Tame and wild matrix problems. In Vlastimil Dlab and Peter Gabriel, editors, *Representation Theory II*, volume 832 of *Lecture Notes in Mathematics*, pages 242–258. Springer Berlin Heidelberg, 1980.

[6] David S. Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[7] Peter Gabriel. Unzerlegbare darstellungen i. *manuscripta mathematica*, 6(1):71–103, 1972.

[8] Peter Gabriel. Finite representation type is open. In Vlastimil Dlab and Peter Gabriel, editors, *Representations of Algebras*, volume 488 of *Lecture Notes in Mathematics*, pages 132–155. Springer Berlin Heidelberg, 1975.

[9] Phillip Griffiths. *Principles of algebraic geometry*. Wiley, New York, 1978.

[10] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[11] B. Huisgen-Zimmermann. A hierarchy of parametrizing varieties for representations. In *Rings, modules and representations*, volume 480 of *Contemp. Math.*, pages 207–239. Amer. Math. Soc., Providence, RI, 2009.

[12] Birge Huisgen-Zimmermann. Classifying representations by way of Grassmannians. *Trans. Amer. Math. Soc.*, 359(6):2687–2719 (electronic), 2007.

[13] Birge Huisgen-Zimmermann. Top-stable degenerations of finite-dimensional representations. I. *Proc. Lond. Math. Soc. (3)*, 96(1):163–198, 2008.

[14] James Humphreys. *Linear algebraic groups*. Springer-Verlag, New York, 1975.